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Robustness of the non-Gibbsian property: some examples

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Abstract. We discuss some examples of measures on lattice systems, which lack the property of being a Gibbs measure in a rather strong sense.

1. Introduction

In recent years extensive research has been done on the occurrence of states (probability measures) on lattice systems which are not of Gibbsian type. Such measures occur for example in renormalization-group studies [8–13, 17, 18, 21, 40], non-equilibrium statistical mechanical models [26, 33, 38, 42], image analysis [5, 15, 34], probabilistic cellular automata [19, 39] and random cluster models [19, 39]. The possibility of their occurrence and their properties have been considered by various authors [1, 7, 14, 20, 22, 24, 28–32, 36, 37, 41, 44]. This non-Gibbsian behaviour has often been considered 'pathological'—undesirable—, and there have been various attempts to control the non-Gibbsianness.

One approach, advocated by Martinelli and Olivieri [36, 37], is to study how the non-Gibbsian measures behave under decimation transformations, that is, to consider the restriction of the measure to some sufficiently sparse periodic sublattice. Various examples where a once renormalized measure is non-Gibbsian have been shown to result in Gibbs measures again after such mappings, mostly, but not exclusively, in the regime where the original model has no phase transition [31, 36, 37].

In another approach, developed by Fernández and Pfister [14], one studies the size of the set of 'pathological' configurations and tries to show that it is small, i.e. of measure zero. In this case one says that the non-Gibbsianness is 'weak' [14, 19, 28, 34].

An even stronger control was recently obtained by Dobrushin in an example first studied in [41]. In this example one considers the restriction of the plus-phase of the two-dimensional Ising model to a one-dimensional sublattice. Here the non-Gibbsian measure can be described as the Gibbs measure for an almost everywhere defined potential [7].

In this paper we present some examples in which the non-Gibbsianness is 'robust', either in the sense of stable under decimations, or in the sense of being due to a large-measure set. It is known [35] that the two notions are not equivalent; indeed, there are examples of measures which have a large set of pathological configurations but which become Gibbsian after decimation.

2. Notation and some standard results

First we will introduce some notation and recall some known facts. For details we refer to [12, 16]. We consider spins placed at the vertices of the *lattice* $\mathbb{Z}^d$. The *configuration*

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space is $\Omega = S^{\mathbb{Z}^d}$, where S is the *single spin space*. The notation ω_Λ for the finite volume projection of $\omega \in \Omega$ to S^Λ will be used. The configuration space will be endowed with its product Borel σ-field $\mathcal{F}$. A product measure χ will be chosen on $(\Omega, \mathcal{F})$ as a *reference measure*. An *interaction* is a family of real valued functions Φ_Λ on S^Λ, indexed by $\mathcal{P}_f(\mathbb{Z}^d)$, the set of finite subsets of $\mathbb{Z}^d$, and with the property $\Phi_\emptyset = 0$. We consider *translation invariant* interactions, i.e. $\Phi_{\Lambda+k}(\omega_{\Lambda+k}) = \Phi_\Lambda(\omega_\Lambda)$, for all $k \in \mathbb{Z}^d$. The interaction Φ is called *absolutely summable* whenever

$$\sum_{\substack{\Lambda \ni 0 \\ \Lambda \in \mathcal{P}_f(\mathbb{Z}^d)}} ||\Phi_\Lambda||_\infty < \infty \tag{2.1}$$

where $||\cdot||_\infty$ denotes the sup-norm. The energy content of a volume Λ is given by the *Hamiltonian*

$$\mathcal{H}^\Phi_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X((\omega \times \xi)_X) \tag{2.2}$$

where $(\omega \times \xi)_x = \omega_x$ if $x \in \Lambda$, and $(\omega \times \xi)_x = \xi_x$ if $x \notin \Lambda$, while $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. Here ξ_{Λ^c} is a particular configuration fixed outside the volume Λ, and plays the role of the *boundary condition*. Whenever the configuration space is compact, absolute summability of the interaction is a natural condition since it guarantees the existence of finite volume Hamiltonians. A probability measure on $(\Omega, \mathcal{F})$ is called a *Gibbs measure* for the interaction Φ at inverse temperature β if a version of its conditional probabilities $\pi_\Lambda(\omega_\Lambda, \xi_{\Lambda^c})$ satisfies the *DLR-equation*:

$$\frac{\pi_\Lambda(\omega_\Lambda, \xi_{\Lambda^c})}{\pi_\Lambda(\tau_\Lambda, \xi_{\Lambda^c})} = \mathrm{e}^{-\beta\{\mathcal{H}^\Phi_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) - \mathcal{H}^\Phi_\Lambda(\tau_\Lambda|\xi_{\Lambda^c})\}} \tag{2.3}$$

for every finite Λ. We denote the collection of these conditional probabilities by $\Pi := \{\pi_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)}$.

We will use the following notion of 'locality' for conditional probabilities: Π is called *quasilocal* if

$$\lim_{\Lambda' \to \mathbb{Z}^d} \sup_{\substack{\xi, \eta \in \Omega \\ \xi_{\Lambda'} = \eta_{\Lambda'}}} |\pi_\Lambda(\cdot, \xi_{\Lambda^c}) - \pi_\Lambda(\cdot, \eta_{\Lambda^c})| = 0 \tag{2.4}$$

for all $\Lambda \subset \Lambda' \in \mathcal{P}_f(\mathbb{Z}^d)$. Π is called *quasilocal at the point* η if

$$\lim_{\Lambda' \to \mathbb{Z}^d} \sup_{\substack{\xi \in \Omega \\ \xi_{\Lambda'} = \eta_{\Lambda'}}} |\pi_\Lambda(\cdot, \xi_{\Lambda^c}) - \pi_\Lambda(\cdot, \eta_{\Lambda^c})| = 0 \tag{2.5}$$

for all $\Lambda \subset \Lambda' \in \mathcal{P}_f(\mathbb{Z}^d)$. For the models we will consider in the following, quasilocality coincides with the continuity of conditional probabilities with respect to the boundary conditions (in the product topology).

Π is said to be *uniformly non-null* with respect to the reference measure χ if, for every $\xi_{\Lambda^c} \in S^{\Lambda^c}$ and $\omega \in \Omega$, there is an $\varepsilon > 0$ such that $\chi_\Lambda(\omega_\Lambda) > 0$ implies $\pi_\Lambda(\omega_\Lambda, \xi_{\Lambda^c}) \geqslant \varepsilon$, for all $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$. (In percolation theory uniform non-nullness is called 'finite energy condition', a terminology which is quite suggestive of a Gibbsian description of the probabilities involved.)

For Gibbs measures the following characterization theorem is known [12, 23, 43].

Theorem 2.1. Let Π be a consistent family of everywhere defined conditional probabilities (a 'specification'), and suppose a reference measure χ is given. The following two statements imply each other.

(1) There exists an absolutely summable interaction Φ such that Π is a family of conditional probabilities corresponding to a Gibbs measure for Φ.

(2) Π is quasilocal, and uniformly non-null with respect to the reference measure χ.

Another useful notion, relating different Gibbs measures, is the relative entropy density. This is defined as follows. Suppose two different probability measures ϱ_1, ϱ_2 are given on $(\Omega, \mathcal{F})$. Denote by $h_{\varrho_1;\varrho_2}$ the Radon–Nikodym derivative of ϱ_1 with respect to ϱ_2, whenever it exists. Suppose, moreover, that $\log h_{\varrho_1;\varrho_2} \in L^1(\varrho_1)$. The quantity

$$I(\varrho_1|\varrho_2) = \begin{cases} \int_\Omega h_{\varrho_1;\varrho_2}(\omega) \log h_{\varrho_1;\varrho_2}(\omega) \varrho_2(\mathrm{d}\omega) & \text{if } \varrho_1 \ll \varrho_2 \\ \infty & \text{otherwise} \end{cases} \tag{2.6}$$

is called the *relative entropy of* ϱ_1 *with respect to* ϱ_2. Denote by $\varrho^{\mathcal{F}_\Lambda}$ the restriction of ϱ to $\mathcal{F}_\Lambda$, the product Borel σ-field for S^Λ. The limit

$$i(\varrho_1|\varrho_2) = \lim_{\Lambda_n \in \mathcal{P}_f(\mathbb{Z}^d)} \frac{1}{|\Lambda_n|} I(\varrho_1^{\mathcal{F}_{\Lambda_n}} | \varrho_2^{\mathcal{F}_{\Lambda_n}}) \tag{2.7}$$

defined in van Hove sense, is called the *relative entropy density* for ϱ_1 with respect to ϱ_2. The relative entropy density is actually the rate function describing the (level-3) large deviation behaviour of ϱ_1 with respect to ϱ_2. However, the limit above need not exist. It is known to exist when ϱ_2 is chosen to be a Gibbs measure, and hence in particular when it is a product measure.

Theorem 2.2. The relative entropy density has the following properties.

(1) $i(\varrho_1|\varrho_2) \geqslant 0$.

(2) Suppose ϱ_1 and ϱ_2 are two Gibbs measures for translation invariant interactions. Then:

(a) $i(\varrho_1|\varrho_2) > 0$ iff ϱ_1 and ϱ_2 are Gibbs measures for different interactions

(b) $i(\varrho_1|\varrho_2) = 0$ iff ϱ_1 and ϱ_2 are Gibbs measures for the same interaction.

For a proof, see for example [16, section 15.3], or [12, section 2.6.6].

Now we turn to considering transformations of Gibbs states. Take a positive integer b, and consider the sublattice $b\mathbb{Z}^d$ having spacing b. This will be the renormalized lattice. In our notation we will not use rescaled distances.

A *renormalization transformation* is a probability kernel T defined by

$$\varrho'(\mathrm{d}\tau) = \int_{S^{\mathbb{Z}^d}} T(\omega, \mathrm{d}\tau) \varrho(\mathrm{d}\omega) \tag{2.8}$$

satisfying the following properties.

(1) The image measure is invariant under a subgroup of the translation group leaving $b\mathbb{Z}^d$ invariant.

(2) It is strictly local in the sense that:

(a) there exist two van Hove sequences $\{\Lambda_n\} \subset \mathcal{P}_f(\mathbb{Z}^d)$ and $\{\Lambda'_n\} \subset \mathcal{P}_f(b\mathbb{Z}^d)$ such that for each $E \in \mathcal{F}_{\Lambda_n}$ the function $T(\cdot, E)$ is $\mathcal{F}'_{\Lambda'_n}$-measurable.

(b) there exists a finite $K > 0$, called *compression factor*, such that

$$\limsup_{n\to\infty} \frac{|\Lambda_n|}{|\Lambda'_n|} \leqslant K = b^d.$$

In the most studied cases the renormalization transformation is a product of kernels defined on blocks of internal spins:

$$T(\omega, \mathrm{d}\tau) = \prod_{x \in b\mathbb{Z}^d} \hat{T}(\omega_{B(x)}, \mathrm{d}\tau_x) \tag{2.9}$$

where $B(x)$ is a block attached to the site x, and $\hat{T}$ is blockwise defined. We will use Ising spin variables $S = \{-1, +1\}$, and take a box $B(x) \subset \mathbb{Z}^d$, a translate of a d-cube such that its first vertex is x. The particular examples of renormalization transformations in which we will be interested in the following are:

- Decimation transformation

$$\hat{T}(\omega_{B(x)}, \mathrm{d}\tau_x) = \delta(\omega_x - \tau_x)\,\mathrm{d}\tau_x \tag{2.10}$$

- Kadanoff transformation with parameter $p > 0$

$$\hat{T}(\omega_{B(x)}, \mathrm{d}\tau_x) = \frac{\exp(p\tau_x \sum_{y\in B(x)} \omega_y)}{2\cosh(p\sum_{y\in B(x)} \omega_y)} \frac{\delta(\tau_x - 1) + \delta(\tau_x + 1)}{2}\,\mathrm{d}\tau_x. \tag{2.11}$$

The decimation transformation is an example of a *deterministic* renormalization transformation while the Kadanoff transformation is an example of a *stochastic* renormalization transformation. Kadanoff transformations with trivial scaling have important applications in image reconstruction problems [5, 15, 17]. For further discussion on renormalization transformations we refer to [12] and references quoted therein.

3. Examples of non-Gibbsianness which are stable under decimation

Consider a massless Gaussian model on $\mathbb{Z}^d$. The configuration space is $\mathbb{R}^{\mathbb{Z}^d}$ and the interaction is defined by

$$\Phi_\Lambda = \begin{cases} \frac{1}{2}V_{jk}(\omega_j - \omega_k) & \text{if } \Lambda = \{j, k\} \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

where $\omega_j, \omega_k \in \mathbb{R}$. The functions V_{jk} are even functions of the differences $\omega_j - \omega_k$, and we assume them to be translation invariant, i.e. $V_{j+l,k+l} = V_{jk}$, for all $j, k, l \in \mathbb{Z}^d$. By particular choices of the potential one can describe in general an *anharmonic crystal*. When all the functions V_{jk} are quadratic, the corresponding system is called a *harmonic crystal*. For harmonic or anharmonic crystals one can ask the question of whether Gibbs measures can be constructed for the given potential (where as reference measure the Lebesgue measure is chosen). It can easily be seen that such a Gibbs measure for a harmonic crystal is actually an example of a *massless Gaussian*, i.e. a probability measure defined by the covariance matrix

$$C_{jk} = \mathrm{cov}(\omega_j, \omega_k) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \hat{c}(q)\,\mathrm{e}^{\mathrm{i}q(j-k)}\,\mathrm{d}q \tag{3.2}$$

with $\hat{c} \in L^1([-\pi, \pi]^d)$, positive and even, and the inverse of the covariance matrix $B_{jk} = C_{jk}^{-1}$ satisfying the massless condition

$$\sum_{k\in\mathbb{Z}^d} B_{jk} = 0. \tag{3.3}$$

The mean of this Gaussian measure we will take to be zero. The link between the harmonic crystal interaction V and the massless Gaussian covariance is given by the relation

$$V_{jk}(\omega) = \tfrac{1}{2}B_{jk}(\omega_j - \omega_k)^2. \tag{3.4}$$

For $d < 3$, such B_{jk} define a long-range interaction, for $d \geqslant 3$ also nearest-neighbour interactions can be obtained. For further details and properties of massless Gaussians we refer to [2–4, 6, 8, 12, 24].

Now we consider the *projected massless Gaussian model* obtained under the map $\omega_j \mapsto \mathrm{sign}\ \omega_j, \forall j$. (Since the set of those configurations for which the sign is zero is negligible, one can choose for this case any value of the projected Gaussian spin variable.) The projected system is thus a system of Ising spins with a probability measure induced by the sign map.

Let us fix a particular Gaussian model which is defined by its covariance matrix. We denote by μ the translation invariant (Gaussian) Gibbs measure with mean zero, and denote the induced measure by ϱ. This measure is known to be a non-Gibbsian measure in any dimension [8, 12, 24]. It is known to remain non-Gibbsian under a general class of deterministic transformations [37]. Our new result is that this remains true for stochastic maps like the Kadanoff transformations. Moreover, we can show that the quasilocality property breaks down for stochastically transformed measures, something which is as yet unknown in the deterministic case.

Theorem 3.1. Consider the Kadanoff transformation K_p with parameter p, and a measure ϱ as defined above. For every $p > 0$, $K_p\varrho$ is non-Gibbsian. In fact, $K_p\varrho$ is not quasilocal.

First we need a lemma [12, 45]:

Lemma 3.2. Suppose ϱ_1 and ϱ_2 are two probability measures on a measurable space $(X, \mathcal{X})$, such that $i(\varrho_1|\varrho_2)$ exists. Consider a renormalization transformation T given on this measure space. Then the relative entropy density $i(T\varrho_1|T\varrho_2)$ exists and

$$i(T\varrho_1|T\varrho_2) \leqslant \text{constant} \times i(\varrho_1|\varrho_2). \tag{3.5}$$

Proof of the theorem. It is known that [3, 8, 12, 24]

$$i(\delta^+|\varrho) = 0 = i(\delta^-|\varrho) \tag{3.6}$$

therefore by the lemma above we have

$$i(K_p\delta^+|K_p\varrho) = 0 \tag{3.7}$$

$$i(K_p\delta^-|K_p\varrho) = 0 \tag{3.8}$$

where δ^+ and δ^- are the Dirac measures on the all-plus and all-minus configurations.

It can be seen by the definition of the Kadanoff transformation that it transforms δ-measures into product measures, thus there exist product measures $\lambda_p^+ \neq \lambda_p^-$ such that

$$K_p\delta^+ = \lambda_p^+ \qquad \forall p \tag{3.9}$$

$$K_p\delta^- = \lambda_p^- \qquad \forall p. \tag{3.10}$$

Since λ_p^+ and λ_p^- are trivially two Gibbs measures for two non-equivalent one-site interactions, and $K_p\varrho$ cannot be a Gibbs measure simultaneously for both of these one-site interactions, by theorem 2.2 we infer that there is no absolutely summable interaction such that $K_p\varrho$ would be a Gibbs measure for it. Furthermore, it is known that the family of conditional probabilities corresponding to the measure ϱ is not uniformly non-null [12, 24], although the measure is strictly positive, that is, every cylinder set has positive measure. Strict positivity is a weaker property than uniform non-nullness, because for uniform non-nullness to hold one needs that each cylinder set has positive measure which remains strictly bounded away from zero under *arbitrary* conditioning. However, it is easy to see that under the Kadanoff map the family of conditional probabilities becomes uniformly non-null, therefore by theorem 2.1 we can conclude that $K_p\varrho$ is not quasilocal. $\square$

Corollary 3.3. Consider an arbitrary decimation transformation T. Then neither the measure $(T \circ K_p)\varrho$, nor the measure $(K_p \circ T)\varrho$ is Gibbsian. This also holds when T is replaced by any finite iterate of T.

Proof. This follows by a similar argument applied to either of the measures by taking note of the fact that a decimation transformation maps a product measure into another product measure, and it maps a Dirac measure into another Dirac measure. (Actually, this applies to a wider class of deterministic transformations.) □

As was shown in [8, 12], some of these projected Gaussians are scaling limits for majority rule transformations, in particular of relevance in high dimensions. Applying a different renormalization-group map to it corresponds in renormalization-group language to making a move in a 'redundant' direction [46]. Such a 'redundant' direction corresponds to taking a coordinate transformation in the (here not existing) space of Hamiltonians.

Remark 3.4. A version of theorem 3.1 remains true for other examples of measures which are strictly positive but not uniformly non-null, in particular for the invariant measures of both the voter model and the Martinelli–Scoppola model.

The *voter model* is an interacting particle system defined by the flip rates

$$c(\omega, x) = \frac{1}{2d} \sum_{y:|x-y|=1} \mathbb{1}_{\{\omega_y \neq \omega_x\}} \tag{3.11}$$

and the variables (the 'voters') ω_x placed on $\mathbb{Z}^d$ can take the values zero and one. It is well known [27] that for $d = 1$ and $d = 2$ the only extremal stationary measures are $\delta_{\mathbf{0}}$ and $\delta_{\mathbf{1}}$, where the notations **0** and **1** correspond to the configurations $\omega_x = 0$ and $\omega_x = 1$, respectively, for all $x \in \mathbb{Z}^d$. For $d \geqslant 3$, however, there is a one-parameter family of extremal stationary translation invariant measures $\{\nu_z\}_{0\leqslant z\leqslant 1}$, parametrized by the density of $\omega_x = 1$ with respect to ν_z. For the voter model, the fact that the relation (3.6) holds for the extremal translation invariant stationary measures ν_z, has been proven for all $d \geqslant 3$ in [26]. It is not known in this case, nonetheless it is believed, that the invariant measures are strictly positive, but the family of conditional probabilities corresponding to them is not uniformly non-null.

The *Martinelli–Scoppola model* [38] is a model with stochastic cluster dynamics on the lattice $\mathbb{Z}^2$. The single spin space is $S = \{0, 1\}$, where $\omega_x = 0$ corresponds to an empty site, and $\omega_x = 1$ corresponds to an occupied site. A maximal connected set of occupied sites is called a cluster. A set $X \subset \mathbb{Z}^2$ is called connected if for any two sites $x, y \in X$ there exists a sequence $\{x_k\}_{k=1,\dots,n} \subset X$ of sites (a path) such that $x_1 = x$, $x_n = y$ and $|x_k - x_{k+1}| = 1, \forall k = 1, \dots, n-1$. The dynamics is defined as follows. At each time t a configuration $\omega^t \in \{0, 1\}^{\mathbb{Z}^2}$ is given. The configuration ω^{t+1} is defined by a process consisting of a simultaneous creation and annihilation operation. The creation operation consists of changing the empty sites at time t into occupied sites at time $t+1$ with probability p at each site independently of other sites. The annihilation operation consists of removing the clusters belonging to the configuration ω^t, independently of each other and with probability 1/2. For sufficiently small probabilities p there is but one invariant measure ϱ for this process. For the Martinelli–Scoppola model the relation (3.6) was proved in [38]. It is not known, but it is suspected, that the family of conditional probabilities of the stationary measure for this model also fails to be uniformly non-null [37]. Since in this model there is no $+/-$ symmetry, $K_p\varrho$ has to be distinguished from a product measure by means of, for example, some correlation functions. Indeed, it is known that there exist some fast decaying non-trivial correlation functions [38].

4. An example of non-quasilocal behaviour on large sets

In this section we show that mixtures of Gibbs measures for different interactions are non-Gibbsian in a rather strong sense. These measures can simply be shown to be non-Gibbsian [12]; here we show that the situation is worse in the sense that every configuration is a point at which quasilocality does not hold.

Let $(\Omega, \mathcal{F}, \chi)$ be a measure space, with $\Omega = S^{\mathbb{Z}^d}$, for some S. Suppose on this measure space ϱ_1 and ϱ_2 are two Gibbs measures for the same interaction at different temperatures β_1 and β_2. It is well known that these two Gibbs measures are singular with respect to each other, or equivalently $||\varrho_1 - \varrho_2||_{\text{var}} = 2$. For notational simplicity we will assume that the interaction is of finite range.

Consider the convex combination $\varrho = \frac{1}{2}(\varrho_1 + \varrho_2)$. Denote by π_Λ, $\pi_\Lambda^{(1)}$ and $\pi_\Lambda^{(2)}$ the conditional probabilities for ϱ, ϱ_1 and ϱ_2, respectively. Then

$$\pi_\Lambda(\cdot, \omega) = \pi_\Lambda^{(1)}(\cdot, \omega) \qquad \text{for } \varrho_1\text{—almost all } \omega \in \Omega \tag{4.1}$$

$$\pi_\Lambda(\cdot, \omega) = \pi_\Lambda^{(2)}(\cdot, \omega) \qquad \text{for } \varrho_2\text{—almost all } \omega \in \Omega \tag{4.2}$$

holds for all finite subsets $\Lambda \subset \mathbb{Z}^d$. We denote by $\mathcal{C}_1$ the set of configurations for which (4.1) holds, and by $\mathcal{C}_2$ the set of configurations for which (4.2) holds. Also, we take the neighbourhood basis

$$\mathcal{U}_{\omega,\Lambda} = \{\omega' : \omega'_\Lambda = \omega_\Lambda\}.$$

Since the two measures are singular with respect to each other, the above considerations lead to the following conclusion.

Theorem 4.1. Consider the sets

$$\mathcal{V}_{\omega,\Lambda}^{(1)} = \mathcal{C}_1 \cap \mathcal{U}_{\omega,\Lambda}$$
$$\mathcal{V}_{\omega,\Lambda}^{(2)} = \mathcal{C}_2 \cap \mathcal{U}_{\omega,\Lambda}.$$

For every $\omega \in \Omega$ there exists a volume $\Lambda' \subset \Lambda$ such that for each two configurations $\xi \in \mathcal{V}_{\omega,\Lambda}^{(1)}$ and $\eta \in \mathcal{V}_{\omega,\Lambda}^{(2)}$, whenever $\text{dist}(\partial\Lambda, \partial\Lambda')$ is larger than the range of the interaction, there is a constant $\varepsilon > 0$, independent of Λ, such that

$$\lim_{\Lambda \to \mathbb{Z}^d} |\pi_{\Lambda'}(\cdot, \xi_{\Lambda^c}) - \pi_{\Lambda'}(\cdot, \eta_{\Lambda^c})| \geqslant \varepsilon.$$

The point here is that the conditional probabilities in Λ' are computed at an inverse temperature β_1 or β_2, according to what happens outside the larger volume Λ, but not depending on the configuration restricted to the annulus between the boundaries of Λ and Λ'. Theorem 4.1 above says that the mixture of two Gibbs measures at different temperatures is non-quasilocal *at every* configuration. This is an example of a measure which fails everywhere to be Gibbsian, thus a case where the 'pathology' is extremely severe. Note that theorem 4.1 can actually be generalized in a straightforward way to any convex combination of two Gibbs states for two non-equivalent interactions. As a side remark, we observe that if the two Gibbs measures both remain Gibbsian under decimation, then the strong non-Gibbsianness of their convex combination is preserved under this decimation.

A particular example of a non-Gibbsian measure for which every configuration is a point of non-quasilocality is provided by the following example. Consider the nearest-neighbour ferromagnetic Ising interaction on the two-dimensional square lattice in the subcritical regime. Denote by μ^+ and μ^- the $+$ phase and the $-$ phase, respectively. In [31] it has been shown that at sufficiently low temperatures the projection to the one-dimensional sublattice $b\mathbb{Z}$ with $b \geqslant 3$ of μ^+ and μ^- are Gibbs measures for two different absolutely summable interactions. Hence we have by theorem 4.1:

Corollary 4.2. The conditional probabilities for a projection of any mixture $\mu = \lambda\mu^+ + (1-\lambda)\mu^-$, $0 < \lambda < 1$, onto $b\mathbb{Z}$, with $b \geqslant 3$, are non-quasilocal at every configuration.

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